# Degenerating variations of Hodge structure in dimension one

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# Lecture 1

# Overall purpose

Describe the theory of degenerating variations of Hodge structure on the punctured disk:

- Focus on the analytic behavior of a polarized VHS.
- Get a conceptual understanding of the linear algebra objects that show up in the limit.
- Simplify the original proofs (by Schmid).

# (Complex) Hodge structures

Let $V$ be a complex vector space.

A Hodge structure of weight $n$ on $V$ is a decomposition

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

A polarization is a hermitian form $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that:

1. The decomposition is orthogonal with respect to $Q$.
2. $(-1)^q Q$ is positive definite on the subspace $V^{p,q}$.

It gives rise to a positive definite hermitian inner product

$$\langle u, v \rangle = \sum_{p+q=n} (-1)^q Q(u^{p,q}, v^{p,q}).$$

# (Complex) Hodge structures

Let $X$ be a compact Kähler manifold. Each

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

is a Hodge structure of weight $n$. In this case, $\langle \alpha, \beta \rangle$ is the inner product on harmonic forms (from the Kähler metric).

As long as we have a polarization, we can describe a Hodge structure by its Hodge filtration

$$F^p V = \bigoplus_{i \geq p} V^{i,n-i}$$

because $V^{p,q} = F^p V \cap (F^{p+1} V)^{\perp}$.

# Variations of Hodge structure

Let $E$ be smooth vector bundle on a complex manifold $X$, together with a flat connection $d \colon A^0(E) \to A^1(E)$.

A variation of Hodge structure (VHS) of weight $n$ on $E$ is a decomposition into smooth subbundles

$$E = \bigoplus_{p+q=n} E^{p,q},$$

such that the flat connection $d$ takes $A^0(E^{p,q})$ into

$$A^{1,0}(E^{p,q}) \oplus A^{1,0}(E^{p-1,q+1}) \oplus A^{0,1}(E^{p,q}) \oplus A^{0,1}(E^{p+1,q-1}).$$

Accordingly, $d = \partial + \theta + \bar{\partial} + \theta^*$; the operator $\theta$ is usually called the Higgs field.

# Variations of Hodge structure

More familiar holomorphic description: $d = d' + d''$

- $d''$ makes $E$ into a holomorphic vector bundle $\mathcal{E}$.
- $d''$ preserves the Hodge bundles

  $$F^p = E^{p,q} \oplus E^{p+1,q-1} \oplus \cdots$$

  and so they give holomorphic subbundles $F^p\mathcal{E}$.
- $d'$ defines a flat connection $\nabla\colon \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$.
- Griffiths transversality $\nabla(F^p\mathcal{E}) \subseteq \Omega_X^1 \otimes F^{p-1}\mathcal{E}$.
- $\bar{\partial}$ makes $E^{p,q}$ into a holomorphic vector bundle $\mathcal{E}^{p,q}$, and one has $\mathcal{E}^{p,q} \cong F^p\mathcal{E}/F^{p+1}\mathcal{E}$.
- The Higgs field is the induced $\mathscr{O}_X$-linear morphism

  $$\mathcal{E}^{p,q} \to \Omega_X^1 \otimes \mathcal{E}^{p-1,q+1}.$$

# Variations of Hodge structure

A polarization of a VHS $E$ is a hermitian pairing

$$Q\colon A^0(E) \otimes_{A^0} \overline{A^0(E)} \to A^0$$

with the following three properties:

1. $Q$ is flat: $dQ(u, v) = Q(du, v) + Q(u, dv)$
2. The decomposition is orthogonal.
3. The expression

$$h(u, v) = \sum_{p+q=k} (-1)^q Q(u^{p,q}, v^{p,q})$$

   defines a positive definite hermitian metric on $E$.

The metric $h$ is called the Hodge metric.

# Plan for the lectures

Study VHS on $\Delta^* = \left\{ t \in \mathbb{C} \,\middle|\, 0 < |t| < 1 \right\}$, especially the behavior of the metric and the Hodge structures near $0 \in \Delta$.

1. Examples
2. Asymptotic behavior of the Hodge metric
3. Asymptotic behavior of the Hodge structures
4. Convergence results

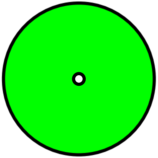
Today: Describe a class of examples where we can understand everything concretely.

- Build intuition
- Models for the general case

# The simplest example

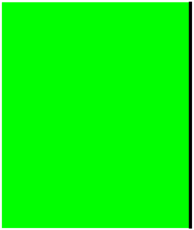
We want to construct a VHS on the punctured disk.

$$\Delta^* = \left\{ t \in \mathbb{C} \;\middle|\; 0 < |t| < 1 \right\} =$$



We work on the universal covering space $\exp\colon \mathbb{H} \to \Delta^*$.

$$\mathbb{H} = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re} z < 0 \right\} =$$



# The simplest example

Consider the trivial bundle $E = \mathbb{H} \times \mathbb{C}^2$; the flat connection $d$ is the usual derivative. The hermitian pairing is constant:

$$Q(e_1, e_1) = Q(e_2, e_2) = 0, \quad Q(e_1, e_2) = 1$$

At the point $z \in \mathbb{H}$, we use the Hodge structure

$$E^{1,0}|_z = \mathbb{C}(e_1 - ze_2), \quad E^{0,1}|_z = \mathbb{C}(e_1 + \bar{z}e_2).$$

This Hodge structure is polarized by $Q$:

$$\begin{aligned} Q(e_1 - ze_2, e_1 + \bar{z}e_2) &= -z + z = 0 \\ (-1)^0 Q(e_1 - ze_2, e_1 - ze_2) &= -z - \bar{z} = 2|\operatorname{Re} z| > 0 \\ (-1)^1 Q(e_1 + \bar{z}e_2, e_1 + \bar{z}e_2) &= 2|\operatorname{Re} z| > 0 \end{aligned}$$

# The simplest example

Consider the trivial bundle $E = \mathbb{H} \times \mathbb{C}^2$; the flat connection $d$ is the usual derivative. The hermitian pairing is constant:

$$Q(e_1, e_1) = Q(e_2, e_2) = 0, \quad Q(e_1, e_2) = 1$$

At the point $z \in \mathbb{H}$, we use the Hodge structure

$$E^{1,0}|_z = \mathbb{C}(e_1 - z e_2), \quad E^{0,1}|_z = \mathbb{C}(e_1 + \bar{z} e_2).$$

It is easy to compute the Hodge decomposition

$$\begin{aligned} e_1 &= \frac{\bar{z}}{z + \bar{z}}(e_1 - z e_2) + \frac{z}{z + \bar{z}}(e_1 + \bar{z} e_2) \\ e_2 &= \frac{-1}{z + \bar{z}}(e_1 - z e_2) + \frac{1}{z + \bar{z}}(e_1 + \bar{z} e_2) \end{aligned}$$

# The simplest example

Let's check that we get a VHS. A smooth section of $E^{1,0}$ looks like $f \cdot (e_1 - ze_2)$, with $f$ smooth.The derivative is

$$df \otimes (e_1 - ze_2) - f\,dz \otimes e_2$$

which can be rewritten as

$$\left(\frac{\partial f}{\partial z}dz + \frac{f\,dz}{z+\bar{z}}\right)\otimes(e_1-ze_2)+\frac{\partial f}{\partial \bar{z}}d\bar{z}\otimes(e_1-ze_2)-\frac{f\,dz}{z+\bar{z}}\otimes(e_1+\bar{z}e_2).$$

This is in $A^{1,0}(E^{1,0}) \oplus A^{0,1}(E^{1,0}) \oplus A^{1,0}(E^{0,1})$.

# The simplest example

We can descend this example to $\Delta^*$. Recall that $t = e^z$.
The deck transformation $z \mapsto z + 2\pi i$ changes the Hodge structures as follows:

$$E^{p,q}\big|_{z+2\pi i} = T \cdot E^{p,q}\big|_z,$$

where $T$ is the matrix

$$T = e^{2\pi i N} = \begin{pmatrix} 1 & 0 \\ -2\pi i & 1 \end{pmatrix}, \quad N = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The quotient of $\mathbb{H} \times \mathbb{C}^2$ by the relation

$$(z, v) \sim (z + 2\pi i, Tv)$$

is a vector bundle on $\Delta^*$, with a polarized VHS of weight 1.

# The simplest example

In the standard basis, the Hodge metric is given by

$$\begin{pmatrix} |x| + y^2|x|^{-1} & -iy|x|^{-1} \\ iy|x|^{-1} & |x|^{-1} \end{pmatrix} \qquad (z = x + iy).$$

The metric grows or decays like powers of $|x| = -\log|t|$.

The behavior is controlled by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

# Representations of $\mathfrak{sl}_2(\mathbb{C})$

We can get other examples from representations of $\mathfrak{sl}_2(\mathbb{C})$; the one above comes from the standard representation.

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is spanned by the matrices

$$\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The relations are $[\mathsf{H}, \mathsf{X}] = 2\mathsf{X}$, $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$, $[\mathsf{X}, \mathsf{Y}] = \mathsf{H}$.

Every finite-dimensional representation $V$ decomposes as

$$V = \bigoplus_{k \in \mathbb{Z}} V_k, \quad V_k = E_k(\mathsf{H})$$

into a sum of weight spaces. They satisfy $\mathsf{X}(V_k) \subseteq V_{k+2}$ and $\mathsf{Y}(V_k) \subseteq V_{k-2}$.

# Representations of $\mathfrak{sl}_2(\mathbb{C})$

The weight spaces are symmetric around $k = 0$:

$$\mathsf{X}^k\colon V_{-k} \xrightarrow{\cong} V_k \quad \text{and} \quad \mathsf{Y}^k\colon V_k \xrightarrow{\cong} V_{-k}.$$

This can also be seen using the Weil element

$$\mathsf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathsf{SL}_2(\mathbb{C}).$$

It has the property that

$$\mathsf{wHw}^{-1} = -\mathsf{H}, \quad \mathsf{wXw}^{-1} = \mathsf{Y}, \quad \text{and} \quad \mathsf{wYw}^{-1} = \mathsf{X}.$$

Moreover, $\mathsf{w}$ induces an isomorphism between $V_k$ and $V_{-k}$.

# Representations of $\mathfrak{sl}_2(\mathbb{C})$

The irreducible representations are $S_m = \mathrm{Sym}^m(\mathbb{C}^2)$, $m \in \mathbb{N}$:

- $S_0 = \mathbb{C}$ is the trivial representation
- $S_1 = \mathbb{C}^2$ is the standard representation

All finite-dimensional representations decompose into irreducible representations, and Schur's lemma gives

$$V \cong \bigoplus_{m \in \mathbb{N}} S_m \otimes \mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}.$$

# $\mathfrak{sl}_2$-Hodge structures

An $\mathfrak{sl}_2$-Hodge structure of weight $n$ on a $\mathbb{C}$-vector space $V$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on $V$ such that:

1. Each weight space $V_k = E_k(\mathrm{H})$ has a Hodge structure of weight $n + k$.
2. Both $\mathrm{X}\colon V_k \to V_{k+2}(1)$ and $\mathrm{Y}\colon V_k \to V_{k-2}(-1)$ are morphisms of Hodge structure.

Concretely, $\mathrm{X}(V_k^{p,q}) \subseteq V_{k+2}^{p+1,q+1}$ and $\mathrm{Y}(V_k^{p,q}) \subseteq V_{k-2}^{p-1,q-1}$.

The typical example is the cohomology of an $n$-dimensional compact Kähler manifold $(X, \omega)$. Here $V_k = H^{n+k}(X, \mathbb{C})$, and

$$\mathrm{X} = 2\pi i\, L_\omega \quad \text{and} \quad \mathrm{Y} = (2\pi i)^{-1}\Lambda_\omega$$

are the Lefschetz operator and its adjoint.

# $\mathfrak{sl}_2$-Hodge structures

A polarization of an $\mathfrak{sl}_2$-Hodge structure $V$ is a hermitian form $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that:

1. $Q$ is nondegenerate and $\mathsf{H}^\dagger = -\mathsf{H}$, $\mathsf{X}^\dagger = \mathsf{X}$, $\mathsf{Y}^\dagger = \mathsf{Y}$.
2. The hermitian form $Q(\_, w\_)$ polarizes the Hodge structure of weight $n + k$ on each weight space $V_k$.

Here the Weil element $\mathsf{w}$ functions as a linear algebra version of the Hodge $*$-operator.

In fact, one checks that $\mathsf{w}\colon V_k \to V_{-k}(-k)$ is an isomorphism of Hodge structures (of weight $n + k$).

# $\mathfrak{sl}_2$-Hodge structures

What does the polarization condition mean concretely?

On the primitive subspace

$$V_{-k} \cap \ker \mathsf{Y} = \ker\Big(\mathsf{X}^{k+1} \colon V_{-k} \to V_{k+2}\Big),$$

the Weil element acts as $\mathrm{w}(v) = \frac{1}{k!}\mathsf{X}^k v$.

Then $Q$ is a polarization exactly if $Q(\_, \mathsf{X}^k\_)$ polarizes the Hodge structure on each primitive subspace $V_{-k} \cap \ker \mathsf{Y}$.

In the Kähler example, this amounts to the Hodge-Riemann bilinear relations.

# $\mathfrak{sl}_2$-Hodge structures

Each irreducible representation $S_m$ has an (essentially unique) $\mathfrak{sl}_2$-Hodge structure of weight $m$:

1. Take $S_0 = \mathbb{C}^{0,0}$ with the hermitian pairing $(u, v) \mapsto u\bar{v}$.
2. For the standard representation on $S_1 = \mathbb{C}^2$, we declare that $e_1$ has Hodge type $(1, 1)$, and $e_2$ has type $(0, 0)$. Our usual pairing

$$Q(e_1, e_1) = Q(e_2, e_2) = 0, \quad Q(e_1, e_2) = 1$$

is a polarization because

$$(-1)^1 Q(e_1, \mathrm{w}e_1) = -Q(e_1, -e_2) = 1$$
$$(-1)^0 Q(e_2, \mathrm{w}e_2) = Q(e_2, e_1) = 1$$

# $\mathfrak{sl}_2$-Hodge structures

Each irreducible representation $S_m$ has an (essentially unique) polarized $\mathfrak{sl}_2$-Hodge structure of weight $m$:

3. For $m \geq 2$, the symmetric power $S_m = \mathrm{Sym}^m(\mathbb{C}^2)$ is spanned by $e_1^m, e_1^{m-1}e_2, \ldots, e_2^m$, and these vectors have Hodge type $(m, m), (m-1, m-1), \ldots, (0, 0)$.

An arbitrary (polarized) $\mathfrak{sl}_2$-Hodge structure of weight $n$ then decomposes as

$$V \cong \bigoplus_{m \in \mathbb{N}} S_m \otimes \mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}.$$

Each vector space $\mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$ inherits a (polarized) Hodge structure of weight $n - m$.

# The associated Hodge structure

## Lemma

*Let $V$ be an $\mathfrak{sl}_2$-Hodge structure of weight $n$, and $Q$ a polarization. Consider the filtration*

$$F^p = \bigoplus_{i \geq p, j} V^{i,j}_{i+j-n}.$$

*Then the following is true:*

1. *The filtration $e^{\mathsf{Y}} F$ is the Hodge filtration of a Hodge structure of weight $n$, polarized by $Q$.*
2. *With respect to the inner product, $\mathsf{H}^* = \mathsf{H}$ and $\mathsf{X}^* = \mathsf{Y}$.*

# The associated Hodge structure

Because of the decomposition

$$V \cong \bigoplus_{m \in \mathbb{N}} S_m \otimes \mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$$

and by functoriality, we only need to check this for $S_1$.

For $S_1 = \mathbb{C}^2$, we get $F^1 = \mathbb{C}e_1$, and so $e^{\mathrm{Y}} F^1 = \mathbb{C}(e_1 + e_2)$. So we get the Hodge structure

$$\mathbb{C}^2 = \mathbb{C}(e_1 + e_2) \oplus \mathbb{C}(e_1 - e_2)$$

from our earlier example (at $z = -1$). As $e_1$ and $e_2$ form an orthonormal basis for the inner product, $\mathrm{H}^* = \mathrm{H}$ and $\mathrm{X}^* = \mathrm{Y}$.

# The associated Hodge structure

From the Hodge structure on $V$, we get a Hodge structure

$$\mathrm{End}(V) = \bigoplus_{j \in \mathbb{Z}} \mathrm{End}(V)^{j,-j}$$

of weight 0 on $\mathrm{End}(V)$. Here

$$\mathrm{End}(V)^{j,-j} = \left\{ A \in \mathrm{End}(V) \;\middle|\; A(V^{p,q}) \subseteq V^{p+j,q-j} \right\}.$$

Write the Hodge decomposition of $A \in \mathrm{End}(V)$ as $A = \sum_j A_j$.

**Lemma**

*In the Hodge structure on* $\mathrm{End}(V)$, *one has*

$$\mathrm{Y} = \mathrm{Y}_{-1} + \mathrm{Y}_0 + \mathrm{Y}_1, \quad \mathrm{X} = -\mathrm{Y}_{-1} + \mathrm{Y}_0 - \mathrm{Y}_1, \quad \mathrm{H} = -2\mathrm{Y}_{-1} + 2\mathrm{Y}_1.$$

Exercise: Check this in the case of $S_1$.

# The associated variation of Hodge structure

Each polarized $\mathfrak{sl}_2$-Hodge structure determines a polarized VHS of the same weight on the punctured disk.

Let $V$ be an $\mathfrak{sl}_2$-Hodge structure of weight $n$.

- Consider the trivial bundle $E = \mathbb{H} \times V$, with flat connection $d$ given by differentiation.
- A polarization $Q$ defines a flat hermitian pairing on $E$.

We know that $e^{\mathsf{Y}} F$ is the Hodge filtration of a polarized Hodge structure on $V$. At the point $z \in \mathbb{H}$, we now use the polarized Hodge structure whose Hodge filtration is

$$\Phi(z) = e^{-z\mathsf{Y}} F.$$

# The associated variation of Hodge structure

Why does this work?

- Set $z = x + iy$, with $x < 0$.
- From $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$, we get

$$e^{-z\mathsf{Y}} = e^{-iy\mathsf{Y}} e^{|x|\mathsf{Y}} = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\mathsf{H}} e^{\mathsf{Y}} e^{\frac{1}{2}\log|x|\mathsf{H}}$$

- The operator $e^{\frac{1}{2}\log|x|\mathsf{H}}$ preserves the filtration $F$, hence

$$e^{-z\mathsf{Y}} F = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\mathsf{H}} \cdot e^{\mathsf{Y}} F.$$

- Both operators belong to the orthogonal group $G = O(V, Q)$, because $\mathsf{H}^\dagger = -\mathsf{H}$ and $\mathsf{Y}^\dagger = \mathsf{Y}$.
- Therefore they map polarized Hodge structures to polarized Hodge structures.

# The associated variation of Hodge structure

Let us check that we get a VHS. Let

$$V = \bigoplus_{p+q=n} V^{p,q}$$

be the Hodge structure with Hodge filtration $e^{\mathsf{Y}} F$.

The Hodge bundle $E^{p,q}$ is then the image of

$$\begin{aligned} \mathbb{H} \times V^{p,q} &\to \mathbb{H} \times V, \\ (z, v) &\mapsto \left(z, e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\mathsf{H}} v\right). \end{aligned}$$

Any smooth section of $E^{p,q}$ therefore looks like

$$e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\mathsf{H}} \cdot f,$$

where $f \colon \mathbb{H} \to V^{p,q}$ is smooth.

# The associated variation of Hodge structure

Set $g = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\mathsf{H}} \in G$. Then

$$d(g \cdot f) = g\left(-\frac{i}{|x|}\mathsf{Y}f \otimes dy + \frac{1}{2|x|}\mathsf{H}f \otimes dx + df\right)$$

Substituting $\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1$ and $\mathsf{H} = -2\mathsf{Y}_{-1} + 2\mathsf{Y}_1$ and simplifying, we find that

$$d = \partial + \theta + \bar{\partial} + \theta^*$$

has the correct shape. For example:

$$\partial = g \cdot \left(\frac{\partial}{\partial z} - \frac{1}{2|x|}\mathsf{Y}_0\right) \cdot g^{-1} \otimes dz \in A^{1,0}(E^{p,q})$$

$$\theta = g \cdot \left(-\frac{1}{|x|}\mathsf{Y}_{-1}\right) \cdot g^{-1} \otimes dz \in A^{1,0}(E^{p-1,q+1})$$

# The associated variation of Hodge structure

From the formula $\Phi(z) = e^{-z\mathrm{Y}}F$, we see that

$$\Phi(z + 2\pi i) = T \cdot \Phi(z),$$

where $T = e^{-2\pi i \mathrm{Y}}$. Note that this operator again has the form $T = e^{2\pi i N}$, with $N = -\mathrm{Y}$ nilpotent.

If we take the quotient of $\mathbb{H} \times V$ by $(z, v) \sim (z + 2\pi i, Tv)$, we again get a flat bundle on the punctured disk.

Our example therefore descends to a polarized variation of Hodge structure of weight $n$ on the punctured disk.

# The associated variation of Hodge structure

From the fact that $\mathsf{H}^* = \mathsf{H}$ and $\mathsf{X}^* = \mathsf{Y}$, we can derive the following formula for the Hodge metric:

$$\begin{aligned} h(u,v) &= \left\langle e^{\frac{1}{2}\log|x|\mathsf{H}} e^{iy\mathsf{Y}} u, e^{\frac{1}{2}\log|x|\mathsf{H}} e^{iy\mathsf{Y}} v \right\rangle \\ &= \left\langle e^{-iy\mathsf{X}} e^{\log|x|\mathsf{H}} e^{iy\mathsf{Y}} u, v \right\rangle \end{aligned}$$

Here the brackets stand for the inner product in the Hodge structure at $z = -1$. After expanding this, we get

$$h(v,v) = \sum_{k=0}^{\infty} \frac{y^{2k}}{(k!)^2} |x|^{\ell-2k} \|\mathsf{Y}^k v\|^2 = |x|^{\ell} \|v\|^2 + \cdots,$$

for a vector $v \in V_\ell$. The growth or decay of the Hodge norm is therefore again controlled by the operator H.

# Lecture 2

# References

I should have said this last time:

- Wilfried Schmid, *Variation of Hodge Structure: The Singularities of the Period Mapping* (Inventiones, 1973)
- Claude Sabbah and Christian Schnell, *Degenerating complex variations of Hodge structure in dimension one*

The first paper is the original source.

In the paper with Claude, we prove the same results, but for complex VHS, and from a more analytic point of view.

# Plan for today

Consider a variation of Hodge structure

$$E = \bigoplus_{p+q=n} E^{p,q}$$

on the punctured disk $\Delta^*$, with polarization $Q$.

Recall the definition of the Hodge metric

$$h(u, v) = \sum_{p+q=n} (-1)^q Q(u^{p,q}, v^{p,q}).$$

**Goal:** Understand the behavior of $h$ near $0 \in \Delta$.

# Multivalued flat sections

We need a fixed reference frame in order to compare the inner products on different fibers of the vector bundle $E$.

We use the vector space $V$ of multivalued flat sections.

Recall the universal covering space

$$\exp\colon \mathbb{H} = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re} z < 0 \right\} \to \Delta^*, \quad z \mapsto e^z.$$

Let $V$ be the space of flat sections of $\exp^*(E, d)$. Then

$$\exp^* E \cong \mathbb{H} \times V.$$

We define the monodromy transformation $T \in \mathrm{GL}(V)$ by

$$(Tv)(z) = v(z - 2\pi i).$$

Then $E$ is the quotient of $\mathbb{H} \times V$ by $(z, v) \sim (z + 2\pi i, Tv)$.

# Multivalued flat sections

The polarization gives us a hermitian form $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ that is nondegenerate and satisfies

$$Q(Tu, Tv) = Q(u, v) \text{ or equivalently } T^{\dagger} T = \mathrm{id}.$$

We have the Jordan decomposition

$$T = T_s \cdot T_u = T_s \cdot e^{2\pi i N},$$

with $T_s \in \mathrm{GL}(V)$ semisimple and $N \in \mathrm{End}(V)$ nilpotent. We note that $T_s$ and $N$ commute and satisfy

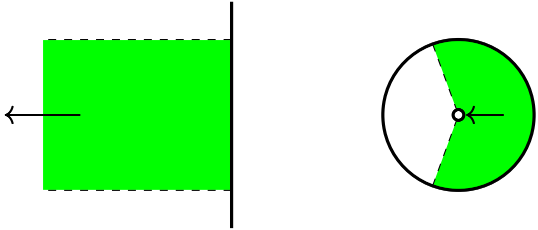
$$T_s^{\dagger} T_s = \mathrm{id} \quad \text{and} \quad N^{\dagger} = N.$$

# Hodge norm estimates

For each nonzero $v \in V$, we get a smooth function

$$h(v, v)\colon \mathbb{H} \to (0, \infty)$$

from the Hodge metric. We want to understand its behavior as $|\operatorname{Re} z| \to \infty$ (which is the same as $t = e^z \to 0$).



# Hodge norm estimates

The Hodge norm always behaves as in the examples from last time: if $v \in V$ is a nonzero multivalued flat section, then

$$h(v, v) \sim |\mathrm{Re}\, z|^k$$

The exponent is controlled by the "weight filtration" of $N$.

## Hodge norm estimates

There is an increasing filtration $W_\bullet = W_\bullet V$ such that

$$v \in W_k \setminus W_{k-1} \quad \Longleftrightarrow \quad h(v, v) \sim |\mathrm{Re}\, z|^k$$

as long as $\mathrm{Im}\, z$ remains bounded. The filtration $W_\bullet$ can be computed from the nilpotent operator $N$.

# Weight filtration

Any nilpotent endomorphism $N \in \mathrm{End}(V)$ determines an increasing filtration $W_\bullet$ on $V$, called the weight filtration.

For a Jordan block, say of size $4 \times 4$:

$$N = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{cases} e_1 \in W_3 \\ e_2 \in W_1 \\ e_3 \in W_{-1} \\ e_4 \in W_{-3} \end{cases}$$

If $V$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$, then the weight filtration of the nilpotent operator $N = \pm \mathrm{Y}$ is

$$W_k = \bigoplus_{\ell \leq k} E_\ell(\mathrm{H}).$$

# Weight filtration

In general, the weight filtration of $N \in \mathrm{End}(V)$ is

$$W_k = \sum_{j \in \mathbb{N}} N^j \left( \ker N^{k+2j+1} \right).$$

It is uniquely determined by two conditions:

1. $N(W_\bullet) \subseteq W_{\bullet - 2}$
2. $N^k \colon \mathrm{gr}_k^W \to \mathrm{gr}_{-k}^W$ is an isomorphism for $k \geq 1$.

Here $\mathrm{gr}_k^W = W_k / W_{k-1}$.

# Flat sections have bounded Hodge norm

For example, consider a flat section of $(E, d)$ on $\Delta^*$.

- Its pullback to $\mathbb{H}$ gives us $v \in V$ with $Tv = v$.
- $Tv = v$ implies that $Nv = 0$.
- We have $\ker N \subseteq W_0$.
- The Hodge norm estimates imply that $h(v, v)$ remains bounded as $|\operatorname{Re} z| \to \infty$.

The conclusion is that the Hodge norm of a flat section is bounded near $0 \in \Delta$.

This is the most important case of the Hodge norm estimates. We will actually prove this directly!

# Outline of the proof

The rest of the lecture is about the proof.

1. Computations with harmonic bundles, universal bound for the Higgs field $\theta$.
2. Special case: boundedness for flat sections
3. General case: comparison with examples

# Harmonic bundles

Let $(E, d)$ be a flat bundle on $\Delta^*$. Given a polarized VHS

$$E = \bigoplus_{p+q=n} E^{p,q},$$

one has a decomposition $d = \partial + \theta + \bar{\partial} + \theta^*$, where

$$\begin{aligned}
\partial &\colon A^0(E^{p,q}) \to A^{1,0}(E^{p,q}) \\
\bar{\partial} &\colon A^0(E^{p,q}) \to A^{0,1}(E^{p,q}) \\
\theta &\colon A^0(E^{p,q}) \to A^{1,0}(E^{p-1,q+1}) \\
\theta^* &\colon A^0(E^{p,q}) \to A^{0,1}(E^{p+1,q-1})
\end{aligned}$$

The operator $\theta$ is called the Higgs field.

# Harmonic bundles

By decomposing $d^2 = 0$, we get the following identities:

$$\partial^2 = \theta^2 = \bar{\partial}^2 = (\theta^*)^2 = 0$$
$$\partial\theta + \theta\partial = \bar{\partial}\theta^* + \theta^*\bar{\partial} = 0$$
$$\bar{\partial}\theta + \theta\bar{\partial} = \partial\theta^* + \theta^*\partial = 0$$
$$\partial\bar{\partial} + \bar{\partial}\partial + \theta\theta^* + \theta^*\theta = 0$$

Since $dQ(u, v) = Q(du, v) + Q(u, dv)$ and different $E^{p,q}$ are orthogonal with respect to $Q$, we also get:

- $\partial + \bar{\partial}$ is a metric connection for $h$.
- $\theta^*$ is the adjoint of $\theta$ relative to $h$.

This means that $(E, d, h)$ is a harmonic bundle (Simpson).

# Harmonic bundles

Let's do some computations with these identities.

**Lemma**

*Let $0 \neq v \in V$ and define $\varphi = \log h(v, v) \colon \mathbb{H} \to \mathbb{R}$.*

1. *$\varphi$ is subharmonic, meaning that $\Delta\varphi \geq 0$.*
2. *We have* $\left|\frac{\partial \varphi}{\partial z}\right| = \left|\frac{\partial \varphi}{\partial \bar{z}}\right| \leq 2 h_{\mathrm{End}(E)}(\theta_{\partial/\partial z}, \theta_{\partial/\partial z})^{1/2}$.

Note that $\theta_{\partial/\partial z}$ is a smooth section of the bundle $\mathrm{End}(E)$.

It maps the subbundle $E^{p,q}$ into $E^{p-1,q+1}$, and in particular, it is nilpotent.

# Harmonic bundles

Let's prove (2), to show the idea. We work on $\mathbb{H}$.

- $\partial + \bar{\partial}$ is a metric connection, and so

$$\partial h(v, v) = h(\partial v, v) + h(v, \bar{\partial} v).$$

- From $dv = 0$, we get $\partial v = -\theta v$ and $\bar{\partial} v = -\theta^* v$, hence

$$\begin{aligned} \partial h(v, v) &= -h(\theta v, v) - h(v, \theta^* v) = -2h(\theta v, v) \\ \frac{\partial}{\partial z} h(v, v) &= -2h(\theta_{\partial/\partial z} v, v). \end{aligned}$$

- The Cauchy-Schwarz inequality then gives

$$\left| \frac{\partial}{\partial z} h(v, v) \right| \leq 2 h_{\mathrm{End}(E)}(\theta_{\partial/\partial z}, \theta_{\partial/\partial z})^{1/2} \cdot h(v, v).$$

# Simpson's basic estimate

The crucial point is that one can bound the norm of $\theta_{\partial/\partial z}$, and therefore the derivative of the function $\varphi = \log h(v, v)$.

**Theorem**

*Let $r = \operatorname{rk} E = \dim V$, and define $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$. Then*

$$h_{\mathrm{End}(E)}(\theta_{\partial/\partial z}, \theta_{\partial/\partial z}) \leq \frac{C_0^2}{|\operatorname{Re} z|^2} \quad \text{for all } z \in \mathbb{H}.$$

The amazing thing is that this only depends on the rank of $E$.

# Simpson's basic estimate

Here is an outline of the proof. Set $A = \theta_{\partial/\partial z}$ and $A^* = \theta^*_{\partial/\partial \bar{z}}$.

**Step 1.** Another calculation with harmonic bundles gives

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h_{\mathrm{End}(E)}(A, A) \geq \frac{h_{\mathrm{End}(E)}\Big([A^*, A], [A^*, A]\Big)}{h_{\mathrm{End}(E)}(A, A)}.$$

Here $[A^*, A]$ is the commutator (as a section of $\mathrm{End}(E)$).

**Step 2.** If $A$ is a nilpotent endomorphism of $V$, and $A^*$ is its adjoint with respect to an inner product, then

$$\|[A^*, A]\|^2 \geq \frac{1}{2C_0^2} \|A\|^4.$$

Applied pointwise, this gives

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h_{\mathrm{End}(E)}(A, A) \geq \frac{1}{2C_0^2} h_{\mathrm{End}(E)}(A, A).$$

# Simpson's basic estimate

**Step 3.** Recall Ahlfors' lemma: For smooth $f : \mathbb{H} \to (0, \infty)$,

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log f \geq \frac{f}{2C} \quad \Longrightarrow \quad f \leq \frac{C}{|\operatorname{Re} z|^2}.$$

**Step 4.** Since we know that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h_{\mathrm{End}(E)}(A, A) \geq \frac{1}{2C_0^2} h_{\mathrm{End}(E)}(A, A),$$

we get the desired inequality

$$h_{\mathrm{End}(E)}(A, A) \leq \frac{C_0^2}{|\operatorname{Re} z|^2}.$$

# Simpson's basic estimate

**Conclusion:** If $0 \neq v \in V$ is a multivalued flat section, then

$$\varphi = \log h(v, v) \colon \mathbb{H} \to \mathbb{R}$$

is subharmonic and

$$\left| \frac{\partial \varphi}{\partial z} \right| = \left| \frac{\partial \varphi}{\partial \bar{z}} \right| \leq \frac{2 C_0}{|\mathrm{Re}\, z|}.$$

# The monodromy theorem

As an exercise, let's prove the monodromy theorem:

If $\lambda \in \mathbb{C}$ is an eigenvalue of $T$, then $|\lambda| = 1$.

- Let $v \in V$ be a nonzero eigenvector with $Tv = \lambda v$.
- Then $v(z - 2\pi i) = \lambda v(z)$, and so $\varphi = \log h(v, v)$ satisfies

$$\log|\lambda|^2 = \varphi(z - 2\pi i) - \varphi(z) = \int_0^1 \frac{d}{dy}\varphi(z - 2\pi iy)\,dy$$

- From the bound on the derivatives of $\varphi$, we get

$$\left|\log|\lambda|^2\right| \leq 4\pi \cdot \frac{2C_0}{|\mathrm{Re}\,z|}.$$

- Letting $|\mathrm{Re}\,z| \to \infty$, we conclude that $|\lambda| = 1$.

# Boundedness of flat sections

Next, let's show that flat sections are bounded.

**Lemma**

*Let $v \in V$ be a multivalued flat section with $Tv = v$. Then the function $h(v, v)$ remains bounded as $|\mathrm{Re}\, z| \to \infty$.*

Consider the function $\varphi = \log h(v, v)$ on $\mathbb{H}$. We know:

1. $\varphi(z + 2\pi i) = \varphi(z)$
2. $\varphi$ is subharmonic: $\Delta\varphi \geq 0$
3. The first derivatives of $\varphi$ satisfy

$$\left|\frac{\partial\varphi}{\partial z}\right| = \left|\frac{\partial\varphi}{\partial\bar{z}}\right| \leq \frac{2C_0}{|\mathrm{Re}\, z|}.$$

It follows that $\varphi$ is bounded from above as $|\mathrm{Re}\, z| \to \infty$.

# Boundedness of flat sections

Here is a gist of the proof, in a toy case:

Let $f\colon (-\infty, 0) \to \mathbb{R}$ be a smooth function such that

$$f''(x) \geq 0 \qquad \text{and} \qquad |f'(x)| \leq \frac{C}{|x|}$$

for some $C > 0$. Then $f$ is bounded from above as $x \to -\infty$.

- $f'' \geq 0$ means that $f'$ is increasing.
- Since $\lim_{x \to -\infty} f'(x) = 0$, it follows that $f'(x) \geq 0$.
- Therefore $f$ is itself increasing, and so

$$f(x) \leq f(-1) \quad \text{for } x \leq -1.$$

# Comparison theorem

**Comparison theorem**

Let $E_1$ and $E_2$ be two polarized VHS on the punctured disk. If $(E_1, d_1) \cong (E_2, d_2)$ as flat bundles, then the Hodge metrics $h_1$ and $h_2$ are mutually bounded, up to a constant, as $t \to 0$.

This is an easy consequence:

- The bundle $H = \mathrm{Hom}(E_1, E_2)$ inherits a polarized VHS.
- An isomorphism $f\colon E_1 \to E_2$ of flat bundles gives a single-valued flat section of $H$ such that $Tf = f$.
- By the lemma, $h_H(f, f)$ stays bounded as $t \to 0$.
- Since $h_2\big(f(v), f(v)\big) \leq h_H(f, f) \cdot h_1(v, v)$, we get one inequality; the other follows by symmetry.

# Proof of the Hodge norm estimates

Let me remind you about the main theorem:

### Hodge norm estimates

There is an increasing filtration $W_\bullet = W_\bullet V$ such that

$$v \in W_k \setminus W_{k-1} \quad \Longleftrightarrow \quad h(v, v) \sim |\operatorname{Re} z|^k$$

as long as $\operatorname{Im} z$ remains bounded. Moreover, the filtration $W_\bullet$ is the weight filtration of the nilpotent operator $N$.

Recall that $T = T_s \cdot e^{2\pi i N}$ is the Jordan decomposition of the monodromy transformation $T \in \mathrm{GL}(V)$.

# Proof of the Hodge norm estimates

Now it is fairly easy to prove the Hodge norm estimates:

- Let $E$ be a polarized VHS on the punctured disk.
- By the comparison theorem, all we need is another VHS on $(E, d)$ whose Hodge metric has the desired behavior.
- By putting $T$ into Jordan canonical form, we can assume that $T$ is a single Jordan block; equivalently, $V$ is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.
- Last time, we showed that each irreducible representation has an $\mathfrak{sl}_2$-Hodge structure. We also saw that the Hodge norm of the associated VHS has the correct behavior.

# Lecture 3

# Plan for today

Let $E$ be a polarized VHS on $\Delta^*$. Recall some notation:

- $V$ is the space of multivalued flat sections.
- $T = T_s \cdot e^{2\pi i N} \in \mathrm{GL}(V)$ is the monodromy operator.
- $W_\bullet$ is the weight filtration of $N$.

Yesterday, we proved that, as $|\operatorname{Re} z| \to \infty$, one has

$$v \in W_k \setminus W_{k-1} \quad \Longleftrightarrow \quad h(v, v) \sim |\operatorname{Re} z|^k$$

(as long as $\operatorname{Im} z$ stays in a bounded interval).

**Goal:** Understand the behavior of the Hodge structures.

The non-uniform behavior of the metric prevents the existence of a limit. We will solve this problem by "rescaling".

# The period domain

In order to compare different Hodge structures on $V$, we need to review spaces of polarized Hodge structures.

Fix $V$ and a hermitian form $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$.

The period domain $D$ parametrizes Hodge structures (with fixed Hodge numbers) on $V$ that are polarized by $Q$:

$$o \in D \quad \Longleftrightarrow \quad V = \bigoplus_{p+q=n} V_o^{p,q}$$

We denote the resulting inner product by $\langle u, v \rangle_o$.

# The period domain

The real Lie group $G = O(V, Q)$ acts transitively on $D$:

$$g \cdot o \in D \quad \Longleftrightarrow \quad V = \bigoplus_{p+q=n} V_{g \cdot o}^{p,q} = \bigoplus_{p+q=n} g(V_o^{p,q}).$$

The two inner products are related by the formula

$$\langle gu, gv \rangle_{g \cdot o} = \langle u, v \rangle_o.$$

The Lie algebra of the group $G$ is

$$\mathfrak{g} = \left\{ A \in \mathrm{End}(V) \;\middle|\; A^\dagger = -A \right\},$$

where $A^\dagger$ means the adjoint with respect to $Q$.

# The period domain

Since a polarized Hodge structure is determined by its Hodge filtration, $D$ embeds as an open set into the compact dual $\check{D}$, the space of decreasing filtrations $F^\bullet$ (with $\dim F^p$ fixed).

The complex Lie group $\mathrm{GL}(V)$ acts transitively on $\check{D}$, and $\check{D}$ is a projective complex manifold. $D \subseteq \check{D}$ is open.

Since the Lie algebra of $\mathrm{GL}(V)$ is just $\mathrm{End}(V)$, the tangent space to $\check{D}$ at a point $o \in D$ is therefore

$$T_o\check{D} \cong \mathrm{End}(V)/F^0\,\mathrm{End}(V),$$

where $F^0\,\mathrm{End}(V) = \left\{ A \in \mathrm{End}(V) \;\middle|\; A(F^\bullet) \subseteq F^\bullet \right\}$.

# The period mapping

From the polarized VHS on $E$ on $\Delta^*$, we get a polarized VHS on $\exp^* E \cong \mathbb{H} \times V$. This gives us the period mapping

$$\Phi\colon \mathbb{H} \to D$$

The Hodge structure at the point $z \in \mathbb{H}$ is

$$V = \bigoplus_{p+q=n} V^{p,q}_{\Phi(z)},$$

and the inner product is $\langle u, v \rangle_{\Phi(z)}$.

The period mapping is holomorphic, basically because the Hodge filtration is preserved by the operator $d'' = \bar{\partial} + \theta^*$.

# The period mapping

Recall that $T \in G$. In fact, $\Phi(z + 2\pi i) = T \cdot \Phi(z)$.

Why? For each $z \in \mathbb{H}$, we have an isomorphism

$$\phi_z \colon V \to E_{t=e^z}, \quad v \mapsto v(z).$$

Since $(Tv)(z + 2\pi i) = v(z)$, the following diagram commutes:

$$\begin{tikzcd}
V \arrow[dd, "T"] \arrow[dr, bend left, "\phi_z"] & \\
& E_{t=e^z} \\
V \arrow[ur, bend right, "\phi_{z+2\pi i}"'] &
\end{tikzcd}$$

The way the period mapping is constructed, we get

$$T^{-1}\Phi^p(z + 2\pi i) = T^{-1}\phi_{z+2\pi i}^{-1}\Big(F^p E_t\Big) = \phi_z^{-1}\Big(F^p E_t\Big) = \Phi^p(z).$$

# Rescaling the period mapping

Using our new notation, we can write the Hodge metric as

$$h(v,v)(z) = \|v\|^2_{\Phi(z)}.$$

We want to understand what happens to the Hodge structures $\Phi(z) \in D$ in the limit as $|\operatorname{Re} z| \to \infty$.

They will not converge in general, because of the non-uniform behavior of the Hodge metric:

$$v \in W_k \setminus W_{k-1} \quad \Longleftrightarrow \quad \|v\|^2_{\Phi(z)} \sim |\operatorname{Re} z|^k$$

# Rescaling the period mapping

We should fix this problem by rescaling: chose a complement

$$W_k = V_k \oplus W_{k-1},$$

and then multiply by $|\mathrm{Re}\, z|^{-k/2}$ on the subspace $V_k$.

The complement is needed because of the implied constant:

$$\|v\|^2_{\Phi(z)} \sim |\mathrm{Re}\, z|^k$$

is an abbreviation for

$$C(v)^{-1}|\mathrm{Re}\, z|^k \leq \|v\|^2_{\Phi(z)} \leq C(v)|\mathrm{Re}\, z|^k$$

but the constant $C(v)$ goes to zero as $v$ approaches $W_{k-1}$.

# Rescaling the period mapping

To do this nicely, we pick $H \in \operatorname{End}(V)$ such that:

1. $H$ is semisimple with eigenvalues in $\mathbb{Z}$
2. $W_k = E_k(H) \oplus W_{k-1}$ for every $k \in \mathbb{Z}$.
3. $[H, N] = -2N$ (recall that $N(W_k) \subseteq W_{k-2}$)
4. $H^\dagger = -H$, meaning that $H \in \mathfrak{g}$.
5. $[H, T_s] = 0$

Many such splittings exist, just by linear algebra.

The first three lines imply that we get a representation

$$\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}(V), \quad \rho(\mathsf{H}) = H, \quad \rho(\mathsf{Y}) = -N.$$

The weight spaces $V_k = E_k(\mathsf{H})$ give us $W_k = V_k \oplus W_{k-1}$.

# Rescaling the period mapping

Now we can rescale. If $v \in E_k(H)$, then $\|v\|^2_{\Phi(z)} \sim |\mathrm{Re}\, z|^k$ and

$$|\mathrm{Re}\, z|^{-k/2} v = e^{-\frac{1}{2}\log|\mathrm{Re}\, z| H} v$$

Since $H \in \mathfrak{g}$, we have $e^{-\frac{1}{2}\log|\mathrm{Re}\, z| H} \in G$. Therefore

$$\|v\|^2_{e^{\frac{1}{2}\log|\mathrm{Re}\, z| H}\Phi(z)} = \left\| e^{-\frac{1}{2}\log|\mathrm{Re}\, z| H} v \right\|^2_{\Phi(z)} = |\mathrm{Re}\, z|^{-k} \|v\|^2_{\Phi(z)}$$

stays bounded as $|\mathrm{Re}\, z| \to \infty$.

But we still have the restriction that $\mathrm{Im}\, z$ needs to lie in a bounded interval. We can get rid of that as follows.

# Rescaling the period mapping

The problem is caused by the fact that $\Phi(z + 2\pi i) = T\Phi(z)$. All eigenvalues of $T$ satisfy $|\lambda| = 1$. Taking their logarithms, we can find a semisimple operator $S \in \mathrm{End}(V)$, with real eigenvalues in an interval of length $< 1$, such that

$$T = T_s e^{2\pi i N} = e^{2\pi i (S+N)}.$$

Then $S^\dagger = S$, and so $e^{-i \operatorname{Im} z (S+N)} \in G$. The expression

$$e^{-i \operatorname{Im} z (S+N)} \Phi(z) \in D$$

is now invariant under the substitution $z \mapsto z + 2\pi i$.

# Rescaling the period mapping

Combining both operations, we arrive at

$$\hat{\Phi}(z) = \underbrace{e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-i \operatorname{Im} z(S+N)}}_{\text{in the real Lie group } G} \Phi(z) \in D.$$

This is invariant under $z \mapsto z + 2\pi i$, and for every $v \in V$,

$$\|v\|^2_{\hat{\Phi}(z)} = \left\| e^{i \operatorname{Im} z(S+N)} e^{\frac{1}{2}\log|\operatorname{Re} z| H} v \right\|^2_{\Phi(z)}$$

remains bounded as $|\operatorname{Re} z| \to \infty$, uniformly in $\operatorname{Im} z$.
We call the (real analytic) mapping

$$\hat{\Phi} \colon \mathbb{H} \to D, \quad \hat{\Phi}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-i \operatorname{Im} z(S+N)} \Phi(z),$$

the rescaled period mapping. It depends on $H$ and $S$.

# Convergence of the rescaled period mapping

The main result is that the rescaled period mapping converges.

**Theorem**

*The limit* $\lim_{|\mathrm{Re}\, z| \to \infty} \hat{\Phi}(z)$ *exists in the period domain* $D$*.*

This shows that the Hodge metric really controls everything:

- The metric has a simple (but non-uniform) behavior: different power of $|\mathrm{Re}\, z|$.
- After we rescale in order to eliminate the different powers, both the metric and the Hodge structures converge.

# Convergence of the rescaled period mapping

The main result is that the rescaled period mapping converges.

**Theorem**

*The limit* $\lim_{|\mathrm{Re}\, z| \to \infty} \hat{\Phi}(z)$ *exists in the period domain* $D$*.*

There is some additional information. The filtration

$$F = e^N \lim_{|\mathrm{Re}\, z| \to \infty} \hat{\Phi}(z) \in \check{D}$$

satisfies $T_s(F^\bullet) \subseteq F^\bullet$, $H(F^\bullet) \subseteq F^\bullet$, and $N(F^\bullet) \subseteq F^{\bullet - 1}$.

# Convergence of the rescaled period mapping

Recall that an $\mathfrak{sl}_2$-Hodge structure

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

has an associated VHS, with $N = -\mathsf{Y}$, and period mapping

$$\Phi(z) = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\mathsf{H}}(e^{\mathsf{Y}}F) \qquad (z = x + iy)$$

In this case,

$$\hat{\Phi}(z) = e^{\frac{1}{2}\log|x|H} e^{-iyN}\Phi(z) = e^{\mathsf{Y}}F = e^{-N}F$$

is a constant Hodge structure. In particular, $F = e^{N}\hat{\Phi}(z)$ is the Hodge filtration in the $\mathfrak{sl}_2$-Hodge structure.

# The limiting $\mathfrak{sl}_2$-Hodge structure

We will prove the convergence next time (together with the "nilpotent orbit theorem", an important intermediate result).

In the rest of today's lecture, I want to deduce from the convergence the existence of a limiting $\mathfrak{sl}_2$-Hodge structure.

From the splitting $H \in \mathrm{End}(V)$, we get a representation

$$\rho\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(V), \quad \rho(\mathsf{H}) = H, \quad \rho(\mathsf{Y}) = -N.$$

We set $V_k = E_k(\mathsf{H})$, so that

$$V = \bigoplus_{k \in \mathbb{Z}} V_k.$$

We will upgrade this to a polarized $\mathfrak{sl}_2$-Hodge structure.

# $\mathfrak{sl}_2$-Hodge structures (review)

Recall that an $\mathfrak{sl}_2$-Hodge structure of weight $n$ on a $\mathbb{C}$-vector space $V$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on $V$ such that:

1. Each weight space $V_k = E_k(\mathsf{H})$ has a Hodge structure of weight $n + k$.
2. Both $\mathsf{X}\colon V_k \to V_{k+2}(1)$ and $\mathsf{Y}\colon V_k \to V_{k-2}(-1)$ are morphisms of Hodge structure.

Recall that a polarization of an $\mathfrak{sl}_2$-Hodge structure $V$ is a hermitian form $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that:

1. $Q$ is nondegenerate and $\mathsf{H}^\dagger = -\mathsf{H}$, $\mathsf{X}^\dagger = \mathsf{X}$, $\mathsf{Y}^\dagger = \mathsf{Y}$.
2. The hermitian form $Q(\_, w\_)$ polarizes the Hodge structure of weight $n + k$ on each weight space $V_k$.

# The limiting $\mathfrak{sl}_2$-Hodge structure

The pairing $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ has the property that $\mathsf{Y}^\dagger = \mathsf{Y}$ and $\mathsf{H}^\dagger = -\mathsf{H}$ (and therefore also $\mathsf{X}^\dagger = \mathsf{X}$).

The theorem gives us a filtration $F \in \check{D}$ such that

$$e^{\mathsf{Y}} F = \lim_{|\mathrm{Re}\, z| \to \infty} \hat{\Phi}(z) \in D$$

and such that

$$\mathsf{Y}(F^\bullet) \subseteq F^{\bullet - 1}, \quad \mathsf{H}(F^\bullet) \subseteq F^\bullet, \quad T_s(F^\bullet) \subseteq F^\bullet.$$

This is enough for a polarized $\mathfrak{sl}_2$-Hodge structure.

# The limiting $\mathfrak{sl}_2$-Hodge structure

**Theorem**

*The filtration $F$ is the Hodge filtration of an $\mathfrak{sl}_2$-Hodge structure of weight $n$, polarized by $Q$.*
*(And $T_s$ is an endomorphism of the $\mathfrak{sl}_2$-Hodge structure.)*

Concretely, this means that each weight space $V_k$ has a Hodge structure of weight $n+k$, whose Hodge filtration is $F \cap V_k$.

This is a formal consequence of the fact that

1. $e^{\mathrm{Y}} F \in D$
2. $\mathrm{Y}(F^{\bullet}) \subseteq F^{\bullet -1}$.
3. $H(F^{\bullet}) \subseteq F^{\bullet}$

Let me try to explain the main point. Warning: Linear algebra!

# The limiting $\mathfrak{sl}_2$-Hodge structure

Recall that we have a decomposition

$$V \cong \bigoplus_{m \in \mathbb{N}} S_m \otimes \mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}.$$

We know from the first lecture that $S_m = \mathrm{Sym}^m(\mathbb{C}^2)$ has a canonical polarized $\mathfrak{sl}_2$-Hodge structure of weight $m$.

It is therefore enough to show that

$$\mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$$

has a polarized Hodge structure of weight $n - m$.

By functoriality, we only need to consider $V^{\mathfrak{sl}_2(\mathbb{C})}$, the space of $\mathfrak{sl}_2$-invariants.

# The subspace of invariants

**Proposition**

Under the conditions above, the subspace

$$V^{\mathfrak{sl}_2(\mathbb{C})} = \left\{ v \in V \,\middle|\, \mathrm{Y}v = \mathrm{H}v = 0 \right\} = V_0 \cap \ker \mathrm{Y}$$

has a Hodge structure of weight $n$, polarized by $Q$, whose Hodge filtration is induced by the filtration $F$.

From $e^{\mathrm{Y}}F \in D$, we get a polarized Hodge structure

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

We will prove that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is a sub-Hodge structure.
This is enough because $e^{\mathrm{Y}}F \cap V^{\mathfrak{sl}_2(\mathbb{C})} = F \cap V^{\mathfrak{sl}_2(\mathbb{C})}$.

# The subspace of invariants

From $e^{Y} F \in D$, we get a polarized Hodge structure

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

We denote the resulting norm by the symbol $\|v\|$.

We write the Hodge decomposition of $v \in V$ in the form

$$v = \sum_{p} v_p, \quad v_p \in V^{p,n-p},$$

# The subspace of invariants

As usual, we have an induced Hodge structure of weight 0 on

$$\mathrm{End}(V) = \bigoplus_{j \in \mathbb{Z}} \mathrm{End}(V)^{j,-j}.$$

We write the Hodge decomposition of $A \in \mathrm{End}(V)$ in the form

$$A = \sum_j A_j, \quad A_j \in \mathrm{End}(V)^{j,-j}$$

Concretely, this means that $A_j(V^{p,q}) \subseteq V^{p+j,q-j}$.

Recall that $A^\dagger$ is the adjoint with respect to the nondegenerate pairing $Q$. If $A \in \mathrm{End}(V)^{j,-j}$, then $A^\dagger \in \mathrm{End}(V)^{-j,j}$.

# The subspace of invariants

The idea is to analyze the Hodge decomposition of $\mathsf{Y}$ and $\mathsf{H}$.

**Lemma**

1. $\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1$ *and* $\mathsf{Y}_0^\dagger = \mathsf{Y}_0$ *and* $\mathsf{Y}_{-1}^\dagger = \mathsf{Y}_1$
2. $\mathsf{H} = -2\mathsf{Y}_{-1} + \mathsf{H}_0 + 2\mathsf{Y}_1$ *and* $\mathsf{H}_0^\dagger = -\mathsf{H}_0$
3. $2\mathsf{Y}_0 = 4[\mathsf{Y}_{-1}, \mathsf{Y}_1] + [\mathsf{Y}_0, \mathsf{H}_0]$

- We have $\mathsf{Y}^\dagger = \mathsf{Y}$, and therefore $\mathsf{Y}_j^\dagger = \mathsf{Y}_{-j}$.
- Since $\mathsf{Y}(e^{\mathsf{Y}} F^\bullet) \subseteq e^{\mathsf{Y}} F^{\bullet - 1}$, we get $\mathsf{Y}_j = 0$ for $j \leq -2$.
- Therefore also $\mathsf{Y}_j = 0$ for $j \geq 2$, and so

$$\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1.$$

- We also have $\mathsf{Y}_0^\dagger = \mathsf{Y}_0$ and $\mathsf{Y}_{-1}^\dagger = \mathsf{Y}_1$.

# The subspace of invariants

The idea is to analyze the Hodge decomposition of Y and H.

**Lemma**

1. $\mathrm{Y} = \mathrm{Y}_{-1} + \mathrm{Y}_0 + \mathrm{Y}_1$ *and* $\mathrm{Y}_0^\dagger = \mathrm{Y}_0$ *and* $\mathrm{Y}_{-1}^\dagger = \mathrm{Y}_1$
2. $\mathrm{H} = -2\mathrm{Y}_{-1} + \mathrm{H}_0 + 2\mathrm{Y}_1$ *and* $\mathrm{H}_0^\dagger = -\mathrm{H}_0$
3. $2\mathrm{Y}_0 = 4[\mathrm{Y}_{-1}, \mathrm{Y}_1] + [\mathrm{Y}_0, \mathrm{H}_0]$

- The second line follows from $\mathrm{H}(F^\bullet) \subseteq F^\bullet$.
- The third line follows from $2\mathrm{Y} = [\mathrm{Y}, \mathrm{H}]$.

# The subspace of invariants

Now we can prove that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is a sub-Hodge structure of $V$. Take any vector $v \in V^{\mathfrak{sl}_2(\mathbb{C})}$, so that $\mathrm{Y}v = \mathrm{H}v = 0$. Write

$$v = v_p + v_{p+1} + \cdots, \quad v_p \neq 0.$$

By induction, it is enough to show that $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$.

**Goal:** Prove that $\mathrm{Y}v_p = \mathrm{H}v_p = 0$.

From $\mathrm{Y}v = 0$ and $\mathrm{Y} = \mathrm{Y}_{-1} + \mathrm{Y}_0 + \mathrm{Y}_1$, we get

$$\mathrm{Y}_{-1}v_p = 0 \quad \text{and} \quad \mathrm{Y}_{-1}v_{p+1} + \mathrm{Y}_0 v_p = 0.$$

From $\mathrm{H}v = 0$ and $\mathrm{H} = -2\mathrm{Y}_{-1} + \mathrm{H}_0 + 2\mathrm{Y}_1$, we get

$$-2\mathrm{Y}_{-1}v_p = 0 \quad \text{and} \quad -2\mathrm{Y}_{-1}v_{p+1} + \mathrm{H}_0 v_p = 0.$$

# The subspace of invariants

So $Y_{-1}v_p = 0$ and $H_0 v_p = -2Y_0 v_p$.
We can exploit the fact that $Y_0^\dagger = Y_0$ but $H_0^\dagger = -H_0$:

$$\begin{aligned} -Q(2Y_0 v_p, v_p) &= Q(v_p, -2Y_0 v_p) = Q(v_p, H_0 v_p) \\ &= Q(-H_0 v_p, v_p) = Q(2Y_0 v_p, v_p) \end{aligned}$$

Therefore $Q(2Y_0 v_p, v_p) = 0$.
Recall the relation $2Y_0 = 4[Y_{-1}, Y_1] + [Y_0, H_0]$. We get

$$\begin{aligned} 0 &= Q(2Y_0 v_p, v_p) \\ &= 4Q(Y_{-1}Y_1 v_p, v_p) + Q(Y_0 H_0 v_p, v_p) - Q(H_0 Y_0 v_p, v_p) \\ &= 4Q(Y_1 v_p, Y_1 v_p) + Q(H_0 v_p, Y_0 v_p) + Q(Y_0 v_p, H_0 v_p) \\ &= 4Q(Y_1 v_p, Y_1 v_p) - 4Q(Y_0 v_p, Y_0 v_p). \end{aligned}$$

# The subspace of invariants

So $Q(\mathrm{Y}_1 v_p, \mathrm{Y}_1 v_p) - Q(\mathrm{Y}_0 v_p, \mathrm{Y}_0 v_p) = 0$.

Now we use the polarization to prove that $\mathrm{Y} v_p = \mathrm{H} v_p = 0$.

We have $v \in V^{p,q}$, hence $\mathrm{Y}_1 v_p \in V^{p+1,q-1}$, and so

$$\|\mathrm{Y}_1 v_p\|^2 = (-1)^{q-1} Q(\mathrm{Y}_1 v_p, \mathrm{Y}_1 v_p).$$

On the other hand, $\mathrm{Y}_0 v_p \in V^{p,q}$, and so

$$\|\mathrm{Y}_0 v_p\|^2 = (-1)^{q} Q(\mathrm{Y}_0 v_p, \mathrm{Y}_0 v_p).$$

Putting both things together, we get

$$\|\mathrm{Y}_1 v_p\|^2 + \|\mathrm{Y}_0 v_p\|^2 = 0,$$

and therefore $\mathrm{Y}_1 v_p = \mathrm{Y}_0 v_p = 0$, hence also $\mathrm{H}_0 v_p = 0$.

# The subspace of invariants

This shows that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is indeed a sub-Hodge structure of $V$.

## Proposition

Under the conditions above, the subspace

$$V^{\mathfrak{sl}_2(\mathbb{C})} = \left\{ v \in V \;\middle|\; \mathsf{Y}v = \mathsf{H}v = 0 \right\} = V_0 \cap \ker \mathsf{Y}$$

has a Hodge structure of weight $n$, polarized by $Q$, whose Hodge filtration is induced by the filtration $F$.

By functoriality, this is enough to conclude that

$$V \cong \bigoplus_{m \in \mathbb{N}} S_m \otimes \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}.$$

has a polarized $\mathfrak{sl}_2$-Hodge structure of weight $n$.

# Lecture 4

# Plan for today

Last time, we introduced the rescaled period mapping

$$\hat{\Phi} \colon \mathbb{H} \to D, \quad \hat{\Phi}(z) = \underbrace{e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-i \operatorname{Im} z (S+N)}}_{\text{in the real group } G} \Phi(z).$$

This is invariant under $z \mapsto z + 2\pi i$, and for every $v \in V$,

$$\|v\|^2_{\hat{\Phi}(z)} = \left\| e^{i \operatorname{Im} z (S+N)} e^{\frac{1}{2}\log|\operatorname{Re} z| H} v \right\|^2_{\Phi(z)}$$

remains bounded as $|\operatorname{Re} z| \to \infty$, uniformly in $\operatorname{Im} z$.

**Theorem**

*The limit* $\displaystyle\lim_{|\operatorname{Re} z| \to \infty} \hat{\Phi}(z)$ *exists in the period domain* $D$.

Today I want to outline the proof of this central result.

# Plan for today

Recall that we had to make two choices:

1. $H$ is a splitting for the weight filtration $W_\bullet$. It is semisimple with eigenvalues in $\mathbb{Z}$. This was to get

$$W_k = E_k(H) \oplus W_{k-1}.$$

2. $S \in \mathrm{End}(V)$ is a logarithm for $T_s$. It is semisimple with eigenvalues in an interval of length $< 1$. This was to get

$$T = e^{2\pi i(S+N)}.$$

# The untwisted period mapping

Unlike $\Phi\colon \mathbb{H} \to D$, the rescaled period mapping $\hat{\Phi}\colon \mathbb{H} \to D$ is no longer holomorphic. The key intermediate step in the proof is a convergence result for a holomorphic mapping.

Recall that $\Phi(z + 2\pi i) = T\Phi(z) = e^{2\pi i(S+N)}\Phi(z)$. Therefore

$$e^{-z(S+N)}\Phi(z) \in \check{D}$$

is invariant under $z \mapsto z + 2\pi i$. This expression is holomorphic in $z$, but no longer lies in $D$ because $e^{-z(S+N)} \notin G$.

We call the holomorphic mapping

$$\Psi\colon \Delta^* \to \check{D}, \quad \Psi(e^z) = e^{-z(S+N)}\Phi(z),$$

the untwisted period mapping.

# The nilpotent orbit theorem

The key step in the proof is the following holomorphic result.

## Nilpotent orbit theorem

The untwisted period mapping extends holomorphically to

$$\Psi\colon \Delta \to \check{D}.$$

The limit $\Psi(0) \in \check{D}$ satisfies $(S+N)\Psi^{\bullet}(0) \subseteq \Psi^{\bullet-1}(0)$.

The proof uses some actual analysis.

1. Explain how to deduce the convergence of $\hat{\Phi}$.
2. Outline the proof of the nilpotent orbit theorem.

# Convergence of the rescaled period mapping

How does this help with the convergence proof? Let's rewrite

$$\hat{\Phi}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-i \operatorname{Im} z (S+N)} \Phi(z)$$

in terms of $\Psi(e^z) = e^{-z(S+N)}\Phi(z)$. We get

$$\begin{aligned}
\hat{\Phi}(z) &= e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-i \operatorname{Im} z (S+N)} \Phi(z) \\
&= e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-|\operatorname{Re} z|(S+N)} \Psi(e^z) \\
&= e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-|\operatorname{Re} z| N} e^{-|\operatorname{Re} z| S} \Psi(e^z) \\
&= e^{-N} \cdot e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-|\operatorname{Re} z| S} \Psi(e^z).
\end{aligned}$$

In the last step, we used $[H, N] = -2N$ for the identity

$$e^{-\frac{1}{2}\log|\operatorname{Re} z| H} e^{-N} e^{\frac{1}{2}\log|\operatorname{Re} z| H} = e^{-|\operatorname{Re} z| N}.$$

# Convergence of the rescaled period mapping

The conclusion is that

$$\hat{\Phi}(z) = e^{-N} \cdot e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-|\operatorname{Re} z| S} \Psi(e^z).$$

We can prove the convergence in $\check{D}$ as follows:

1. By the nilpotent orbit theorem, $\Psi(e^z)$ converges to $\Psi(0)$ at a rate of $|e^z| = e^{-|\operatorname{Re} z|}$ (because it is holomorphic).
2. The operator $e^{-|\operatorname{Re} z| S}$ is of order $e^{-(1-\varepsilon)|\operatorname{Re} z|}$, because the eigenvalues of $S$ lie in an interval of length $< 1$.
3. The operator $e^{\frac{1}{2}\log|\operatorname{Re} z| H}$ is of order $|\operatorname{Re} z|^m$, because the eigenvalues of $H$ are integers.

This is enough to conclude that $\lim_{|\operatorname{Re} z| \to \infty} \hat{\Phi}(z) \in \check{D}$ exists.

# Convergence of the rescaled period mapping

We now use the following basic lemma.

**Lemma**

*Let $f : \mathbb{N} \to D$ be a sequence of points in $D$ such that:*

1. *The limit $\lim_{m\to\infty} f(m)$ exists in $\check{D}$.*
2. *There is a constant $C > 0$ such that*

$$C^{-1}\|v\|^2 \leq \|v\|^2_{f(m)} \leq C\|v\|^2$$

*for every $v \in V$ (where $\|\_\|$ is a fixed norm on $V$).*

*Then $\lim_{m\to\infty} f(m)$ belongs to $D$.*

This holds for $\hat{\Phi}(z)$ because we rescaled the Hodge metric.

# Filtrations and limits

Last time, we defined

$$F = e^N \lim_{|\mathrm{Re}\, z| \to \infty} \hat{\Phi}(z) = \lim_{|\mathrm{Re}\, z| \to \infty} e^{\frac{1}{2} \log |\mathrm{Re}\, z| H} e^{-|\mathrm{Re}\, z| S} \Psi(0).$$

This is the Hodge filtration of the limiting $\mathfrak{sl}_2$-Hodge structure.

How is this filtration $F \in \check{D}$ related to $\Psi(0) \in \check{D}$?

# Filtrations and limits

**Lemma**

*Let $S \in \mathrm{End}(V)$ be semisimple with real eigenvalues. The limit*

$$F_S = \lim_{x \to \infty} e^{xS} F \in \check{D}$$

*exists for any $F \in \check{D}$, and one has $S(F_S^\bullet) \subseteq F_S^\bullet$, which means that $F_S$ is compatible with the eigenspace decomposition of $S$.*

Concretely, $F_S$ is obtained from $F$ as follows:

$$F_S^p \cap E_\alpha(S) \cong \frac{F^p \cap \bigoplus_{\beta \leq \alpha} E_\beta(S)}{F^p \cap \bigoplus_{\beta < \alpha} E_\beta(S)}$$

In other words, we have to project $F$ to the subquotients of the filtration by increasing eigenvalues of $S$.

# Filtrations and limits

This process is used twice in the computation of

$$F = \lim_{|\mathrm{Re}\, z| \to \infty} e^{\frac{1}{2}\log|\mathrm{Re}\, z| H} e^{-|\mathrm{Re}\, z| S} \Psi(0).$$

Since $T_s = e^{2\pi i S}$, the first limit produces a new filtration

$$F_{lim} = \lim_{|\mathrm{Re}\, z| \to \infty} e^{-|\mathrm{Re}\, z| S} \Psi(0)$$

with the property that $T_s(F^\bullet_{lim}) \subseteq F^\bullet_{lim}$. This is usually called the limiting Hodge filtration.

In the second step, we get $F$ from $F_{lim}$ by projecting to the subquotients of the weight filtration $W_\bullet$ (which is the filtration by increasing eigenvalues of $H$). It satisfies $H(F^\bullet) \subseteq F^\bullet$.

# Filtrations and limits

Last time, we showed that the filtration $F$ is the Hodge filtration of an $\mathfrak{sl}_2$-Hodge structure. In particular, each weight space $E_k(H)$ has a Hodge structure of weight $n+k$.

But $F$ is obtained from $F_{lim}$ by projecting to the subquotients of the weight filtration $W_\bullet$. Because of the isomorphism

$$E_k(H) \cong \mathrm{gr}_k^W = W_k/W_{k-1},$$

we see that each $\mathrm{gr}_k^W$ has a Hodge structure of weight $n+k$, whose Hodge filtration is induced by $F_{lim}$.

In other words, we get a mixed Hodge structure on $V$ whose Hodge filtration is $F_{lim}$ and whose weight filtration is $W_{\bullet-n}$. Without any choices, the $\mathfrak{sl}_2$-Hodge structure lives on

$$\bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^W$$

# Proof of the nilpotent orbit theorem

In the remaining time, I would like to describe the proof of the nilpotent orbit theorem.

## Nilpotent orbit theorem

The untwisted period mapping extends holomorphically to

$$\Psi \colon \Delta \to \check{D}.$$

The limit $\Psi(0) \in \check{D}$ satisfies $(S + N)\Psi^{\bullet}(0) \subseteq \Psi^{\bullet-1}(0)$.

Important ingredients:

1. Curvature of the Hodge metric.
2. $L^2$-estimates for the $\bar{\partial}$-equation.
3. Differential equations with regular singular points.

# Curvature of the Hodge metric

Let $(E, d)$ be a flat bundle on $\Delta^*$. Given a polarized VHS

$$E = \bigoplus_{p+q=n} E^{p,q},$$

one has a decomposition $d = \partial + \theta + \bar{\partial} + \theta^*$. These operators satisfy many identities, and so $(E, d, h)$ is a harmonic bundle.

The identities we are going to need today are:

1. $\partial + \bar{\partial}$ is a metric connection for $h$
2. $\theta^*$ is the adjoint of $\theta$ relative to $h$
3. $\partial\bar{\partial} + \bar{\partial}\partial + \theta\theta^* + \theta^*\theta = 0$
4. $\bar{\partial}\theta + \theta\bar{\partial} = 0$
5. $\partial^2 = \bar{\partial}^2 = 0$

# Curvature of the Hodge metric

The operator $\bar{\partial}\colon A^0(E^{p,q}) \to A^{0,1}(E^{p,q})$ makes $E^{p,q}$ into a holomorphic vector bundle. Since $\partial + \bar{\partial}$ is a metric connection, we can compute the curvature of the Hodge metric:

$$\begin{aligned} h(\Theta u, u) &= h\Big((\partial\bar{\partial} + \bar{\partial}\partial)u, u\Big) = -h\Big((\theta\theta^* + \theta^*\theta)u, u\Big) \\ &= h(\theta u, \theta u) + h(\theta^* u, \theta^* u) \end{aligned}$$

for any smooth section $u \in A^0(E^{p,q})$, and therefore

$$h(\Theta_{\partial/\partial t \wedge \partial/\partial \bar{t}} u, u) = h(\theta_{\partial/\partial t} u, \theta_{\partial/\partial t} u) - h(\theta^*_{\partial/\partial \bar{t}} u, \theta^*_{\partial/\partial \bar{t}} u).$$

The curvature tensor is neither positive nor negative.

We can fix this problem as follows. If we multiply $h$ by $e^{-\varphi}$, then the curvature tensor changes to $\Theta + \partial\bar{\partial}\varphi$.

# Curvature of the Hodge metric

From the basic estimate for the Higgs field, we get

$$h(\theta^*_{\partial/\partial\bar{t}}u, \theta^*_{\partial/\partial\bar{t}}u) \leq \frac{C_0^2}{|t|^2(-\log|t|)^2} h(u,u),$$

Now if we set $e^{-\varphi} = |t|^a(-\log|t|)^b$, then $\partial\bar{\partial}\varphi = \frac{b/4}{|t|^2(-\log|t|)^2}$.
The conclusion is that a metric of the form

$$h \cdot |t|^a(-\log|t|)^b$$

will have positive curvature for $b \gg 0$ (Cornalba-Griffiths).
This is the crucial point that makes everything work.

# Hörmander's $L^2$-estimates in one dimension

Let $E$ be a smooth vector bundle on a domain $\Omega \subseteq \mathbb{C}$, with holomorphic structure $d''\colon A^0(E) \to A^{0,1}(E)$. Given $f \in A^0(\Omega, E)$, we want to solve the $\bar{\partial}$-equation $d''_{\partial/\partial\bar{t}} u = f$.

Suppose $E$ has a hermitian metric $h$ with positive curvature: there is a positive function $\rho$ such that

$$h\Big(\Theta_{\partial/\partial t \wedge \partial/\partial\bar{t}}\, \alpha, \alpha\Big) \geq \rho^2 h(\alpha, \alpha) \quad \text{for all } \alpha \in A^0_c(\Omega, E).$$

Under these assumptions, there is a solution $u \in A^0(\Omega, E)$ to the $\bar{\partial}$-equation $d''_{\partial/\partial\bar{t}} u = f$ that satisfies the $L^2$-estimate

$$\int_\Omega h(u, u) d\mu \leq \int_\Omega \frac{1}{\rho^2} h(f, f) d\mu,$$

provided that the right-hand side is finite.

# Proof of the nilpotent orbit theorem

We return to our problem: we want to show that

$$\Psi(e^z) = e^{-z(S+N)}\Phi(z)$$

extends holomorphically across $0 \in \Delta$.

We have an induced VHS of weight 0 on the bundle

$$\mathrm{End}(E) = \bigoplus_j \mathrm{End}(E)^{j,-j}.$$

Viewed as a section of $\mathrm{End}(E)^{-1,1}$, with holomorphic structure $[\bar{\partial}, \_]$, the Higgs field $\theta_{\partial/\partial t}$ is holomorphic:

$$[\bar{\partial}, \theta] = \bar{\partial}\theta + \theta\bar{\partial} = 0$$

But viewed as a section of $\mathrm{End}(E)$, with holomorphic structure $[d'', \_]$, it is not holomorphic:

$$[d'', \theta] = [\bar{\partial} + \theta^*, \theta] = [\theta^*, \theta] \neq 0.$$

# Proof of the nilpotent orbit theorem

**Step 1.** We lift $t\theta_{\partial/\partial t}$ to a holomorphic section $\vartheta$ of the Hodge bundle $F^{-1}\operatorname{End}(E)$, such that

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0 \operatorname{End}(E).$$

Hörmander's $L^2$-estimates let us do this in such a way that

$$\int_{\Delta^*} h_{\operatorname{End}(E)}(\vartheta, \vartheta)|t|^a(-\log|t|)^b d\mu < +\infty.$$

This works because $f = [d''_{\partial/\partial \bar{t}}, t\theta_{\partial/\partial t}] = t[\theta^*_{\partial/\partial \bar{t}}, \theta_{\partial/\partial t}]$ satisfies

$$h(f, f) \leq 2|t|^2 \left( \frac{C_0^2}{|t|^2(-\log|t|)^2} \right)^2 = \frac{2C_0^4}{|t|^2(-\log|t|)^4},$$

by the basic estimate for the Higgs field from Lecture 2.

# Proof of the nilpotent orbit theorem

**Step 2.** Pulling back to $\mathbb{H}$, we get a holomorphic mapping

$$\vartheta\colon \mathbb{H} \to \mathrm{End}(V)$$

with $\vartheta(z + 2\pi i) = T\vartheta(z)T^{-1}$. Since $t = e^z$, this satisfies

$$\vartheta \equiv \theta_{\partial/\partial z} \mod F^0 \,\mathrm{End}(V)_{\Phi(z)}.$$

Untwisting gives us a holomorphic mapping

$$B\colon \Delta^* \to \mathrm{End}(V), \quad B(e^z) = e^{-z(S+N)}\vartheta(z)e^{z(S+N)}.$$

For suitable $a > -2$ and $b \gg 0$, the $L^2$-estimate implies that $B$ is square integrable around the origin.

Therefore $B$ extends holomorphically to $\Delta$.

# Proof of the nilpotent orbit theorem

**Step 3.** The tangent space to $\check{D}$ at the point $\Phi(z) \in D$ is

$$T_{\Phi(z)} \check{D} \cong \operatorname{End}(V) / F^0 \operatorname{End}(V)_{\Phi(z)}.$$

The derivative of the period mapping $\Phi \colon \mathbb{H} \to \check{D}$ is

$$\theta_{\partial/\partial z} \mod F^0 \operatorname{End}(V)_{\Phi(z)}.$$

Therefore the derivative of $z \mapsto \Psi(e^z) = e^{-z(S+N)} \Phi(z)$ is

$$\begin{aligned} e^{-z(S+N)} \theta_{\partial/\partial z} e^{z(S+N)} - (S+N) \\ \equiv B(e^z) - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi(e^z)}. \end{aligned}$$

The operator on the right-hand side is holomorphic!

# Proof of the nilpotent orbit theorem

**Step 4.** Let $g \colon \mathbb{H} \to \mathrm{GL}(V)$ be the unique (holomorphic) solution of the initial value problem

$$g'(z) = \Big(B(e^z) - (S+N)\Big) \cdot g(z), \quad g(-1) = \mathrm{id}\,.$$

Then the derivative of $g(z)^{-1}\Psi(e^z)$ vanishes, and so the mapping $g(z)^{-1}\Psi(e^z)$ is constant. This means that

$$\Psi(e^z) = g(z) \cdot \Psi(e^{-1}).$$

# Proof of the nilpotent orbit theorem

**Step 5.** The differential equation

$$g'(z) = \Big(B(e^z) - (S + N)\Big) \cdot g(z)$$

has a regular singular point at $t = 0$. By the basic theory of such equations, the solution has the form

$$g(z) = M(e^z) \cdot e^{Az}$$

with $M \colon \Delta^* \to \mathrm{GL}(V)$ meromorphic and $A \in \mathrm{End}(V)$.
Since $\Psi(e^z)$ is single-valued, we get

$$\Psi(t) = M(t) \cdot \Psi(e^{-1}),$$

and because $\check{D}$ is projective, it follows that $\Psi$ extends.

# Proof of the nilpotent orbit theorem

The rest of the proof is about deriving good estimates for the rate of convergence, using the maximum principle.

This is important for extending the theory to several variables.

Thank you!